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The subregular variety to the variety of special lattices

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ABSTRACT

We recall the basic geometric properties of the full lattice variety, the projective variety parametrizing special lattices over Witt vectors which was introduced in Haboush (2005) [6]. It is an analog in unequal characteristic, of a certain Schubert variety in the affine Grassmannian for SL_n, and it is normal and a locally complete intersection (Haboush and Sano, submitted for publication [7], Sano (2004) [15]). In this paper, I prove that the complement of its smooth locus, the subregular variety in it, is also normal and a locally complete intersection. The result is analogous to the geometry of the subregular subvariety of the nilpotent cone.

1. Introduction

In [6], Haboush constructed projective varieties parametrizing special lattices over the ring of Witt vectors. The ind-variety which is the direct limit of these varieties is an analog of the affine Grassmannian in unequal characteristic. Let p be a prime, k the algebraic closure of the finite field $\mathbb{F}_p$, $\mathcal{O} = \mathfrak{W}(k)$ the k-points of the Witt vectors, and K the fraction field of $\mathcal{O}$. As in [6], we view K as an ind-k-scheme of the additive group $(K, +)$. Let $F = \mathcal{O}^n$ be the standard $\mathcal{O}$-submodule of K^n with a K-basis $\{e_1, \dots, e_n\}$. A *lattice* L is a free $\mathcal{O}$-submodule of rank n in K^n generated by a K-basis; it is called *special* with respect to F if $\bigwedge^n L = \bigwedge^n F$, and it is of *height* at most $r > 0$ if $L \subset p^{-r}F$. Denote by $\mathbb{L}\mathrm{at}^n_r(K)$ the set consisting of all special lattices of height at most r with respect to F.

Let $G_0 = SL_n(\mathcal{O})$ and B_0 be the Iwahori subgroup of G_0 whose subdiagonal entries are divisible by p. Various parts of [6] are summarized in

Theorem 1.1.

(1) *The set $\mathbb{L}\mathrm{at}^n_r(K)$ is a projective k-variety.*
(2) *For every dominant cocharacter $\gamma = \mathrm{diag}(p^{r_1}, \dots, p^{r_n}) \in \Gamma^+$, that is, $(n-1)r \geq r_1 \geq \cdots \geq r_n \geq -r$ and $\sum_{i=1}^n r_i = 0$, the G_0-orbit closures $\overline{G_0\gamma F}$ have even k-dimensions equal to $2\sum_{i=2}^n (1-i)r_i$ (the proalgebraic action defined up to a suitable Frobenius cover of G_0).*
(3) *The smooth locus $\mathbb{L}\mathrm{at}^n_r(K)^{\mathrm{reg}}$ is the orbit $G_0\mu_r F$, where μ_r is the diagonal matrix with respect to the basis $\{e_1, \dots, e_n\}$ with the diagonal entries $p^{(n-1)r}, p^{-r}, \dots, p^{-r}$. In particular, $\dim \mathbb{L}\mathrm{at}^n_r(K) = \dim G_0\mu_r F = (n-1)nr$, and the complement of $G_0\mu_r F$ is of codimension two hence nonsingular in codimension one.*
(4) *The orbit $B_0\mu_r F$ is isomorphic to the affine space $\mathbb{A}_k^{(n-1)nr}$ and its complement is the closure of the unique codimension one B_0-orbit $B_0\delta_r F$ where $\delta_r = \mathrm{diag}(p^{-r}, p^{(n-1)r}, p^{-r}, \dots, p^{-r})$.*

Item (4) is one reason why we would like to consider $\mathbb{L}\mathrm{at}^n_r(K)$ as an analog of an affine Schubert variety in unequal characteristic.

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Remark 1.2. Let T be the maximal split torus of $G(K)$, and N the normalizer in $G(K)$ of $T(\mathcal{O})$. Consider the coset space W_a/W where $W_a = \langle S_a \rangle$ is the affine Weyl group of the Tits system $(G(K), B_0, N, S_a)$ and $W = \langle S \rangle$ is the finite Weyl group, generated respectively by $S_a = \{s_i \mid 0 \le i \le n-1\}$ and $S = \{s_i \mid i \ne 0\}$ (where s_i with $i \ne 0$ are identified with the permutation matrices of interchanging the ith and the $i+1$th positions, and s_0 with $\begin{pmatrix} & & -p^{-1} \\ & I & \\ p & & \end{pmatrix}$). Then one can check that μ_r in W_a/W has the length $\ell(\mu_r) = (n-1)nr$ in the reduced decomposition $\mu_r = (s_1 \cdots s_{n-1} s_0)^{(n-1)r}$. One also has $\ell(\delta_r) = \ell(\mu_r) - 1$ as $\delta_r = s_1 \mu_r s_1^{-1} = s_1 \mu_r$ in W_a/W.

The natural inclusions, for all $s > 0$,

$$p^{(n-1)(r+s)}F \subset p^{(n-1)r}F \subset L \subset p^{-r}F \subset p^{-(r+s)}F$$

gives the closed embeddings

$$\mathbb{L}\mathrm{at}_r^n(K) \hookrightarrow \mathbb{L}\mathrm{at}_{r+s}^n(K).$$

Write $\varinjlim_r \mathbb{L}\mathrm{at}_r^n(K)$ for the direct limit with the embeddings above. Viewed as an ind-k-scheme structure, this is the space of all special lattices in K^n which is an unequal-characteristic analog of the affine Grassmannian associated to the algebraic loop group for $G = \mathrm{SL}_n$.

In previous works [6,15,7], a simplification was made, when we exclusively consider a fixed r, by replacing the special lattices in $p^{-r}F$ by lattices in F of codimension nr. The set of all such lattices forms a scheme Y_r, which is G_0-isomorphic to $\mathbb{L}\mathrm{at}_r^n(K)$. In the following, we will work with Y_r as the lattice variety.

In [15], the author proved

Theorem 1.3. *The variety Y_r is normal.*

In the recent paper [7], the authors presented a shorter and more canonical proof of the normality, and observed that the basic construction in the proof of Theorem 1.3 carries the result further; namely,

Theorem 1.4. *The variety Y_r is a locally complete intersection.*

As an application of the geometry of Y_r, its Picard group as well as its generator, the canonical line bundle, are computed (cf. [5,7]).

In this paper, we will view the complement of the smooth locus of Y_r, the subregular variety, as a certain Schubert subvariety in Y_r. We shall prove that it is also normal and a locally complete intersection (Theorem 1.5). This result is stronger than the usually expected Cohen–Macaulay property. This is because of very explicit computations that are possible when one limits the study to $G = \mathrm{SL}_n$ and to certain Schubert subvarieties. The methods used here are independent of whether the local field is equal or unequal characteristic, so our proof is also valid for the equal-characteristic analog of these particular Schubert varieties in the affine Grassmannian for SL_n. In proving algebro-geometric properties of Schubert varieties, it seems natural to expect the proofs to closely resemble ones in the equal-characteristic analog, but it seems not the case thus far.

Recall that the well-known partial ordering in Γ of cocharacters that if $\gamma = \mathrm{diag}(p^{r_1}, \ldots, p^{r_n})$ and $\eta = \mathrm{diag}(p^{s_1}, \ldots, p^{s_n})$, then

$$\gamma \preceq \eta$$

if and only if

$$r_1 \le s_1, \quad r_1 + r_2 \le s_1 + s_2, \ldots, \quad \text{and} \quad r_1 + \cdots + r_n = s_1 + \cdots + s_n.$$

For the rest, fix the index r, and for all i such that $0 \le i \le nr$, denote by $\gamma(i)$ the diagonal matrix $\mathrm{diag}(p^{nr-i}, p^i, 1, \ldots, 1)$ so that $\gamma(i) \preceq \mu_r$. Denote by $Y(i)$ the closure of G_0-orbit, $\overline{G_0\gamma(i)F}$ in Y_r. Then $Y_r = Y(0)$ and Y_r^{sub}, the subregular variety of Y_r, is the complement of the smooth locus of Y_r, which by Theorem 1.1, is precisely given by the G_0-orbit closure $Y(1) = \overline{G_0\gamma(1)F}$. We shall call $Y(i)$ the ith *generalized subregular varieties* of Y_r.

Our goal is to establish

Theorem 1.5 (*Main Theorem*)**.** *The subvariety $Y(1)$ of Y_r is normal and a locally complete intersection.*

Whether the remaining $Y(i)$ are locally complete intersections (hence normal, with the known result of regularity in codimension one, see Theorem 1.1) remains unknown.

We discuss below some previous results for various similar cases related to this paper. Over the local field K of equal characteristic, more specifically for $K = k((t))$ with the residue field k of arbitrary characteristic and G a reductive algebraic group, it is known that all affine Schubert varieties in affine (partial) flag varieties for $G(K)$ are normal and Cohen–Macaulay with rational singularities (cf. [4]). In particular, Cohen–Macaulay property is proved using various techniques including Demazure resolutions, their Frobenius splitting, and cohomology vanishing, extending the same techniques in the case of a finite-dimensional flag variety. Over the local field of unequal characteristic, the general theory of affine-Grassmannian-like or affine-flag-like varieties for all reductive groups is yet to emerge.

Possibility still remains, of proving the analogous statement of Cohen–Macaulay and normal properties for *all* Schubert subvarieties of Y_r following the same techniques used in finite-dimensional or equal-characteristic cases, such as defining a suitable Demazure desingularization by iterated $\mathbb{P}^1_k$-bundles. However, the k-scheme-theoretic construction of such a desingularization must be done by incorporating all the morphisms defining k-scheme-theoretic multiplications

$$p^{-r}\mathcal{O}^{(s)} \times p^{-s}\mathcal{O}^{(r)} \to p^{-(r+s)}\mathcal{O}$$

with appropriate Frobenius coverings $X^{(r)} \to X$, purely inseparable morphisms. This is necessary in the group-scheme-theoretic construction with linear groups whose matrix entries allow various negative prime powers. This is one obstruction faced by, for example, the construction of Demazure desingularization in general, since the birational morphism from the desingularization is given by various multiplication morphisms.

Instead, the proofs of Theorems 1.3–1.5 rely on the passage of the locally complete intersection property of a covering scheme under a smooth (more generally, a faithfully flat) morphism (cf. [1]) onto Y_r. This approach is similar to the proof of the normality of the nilpotent variety (full nilpotent cone) in [9], and the methods in the paper also apply to the same class of Schubert varieties in the equal-characteristic analog. Works such as [10,13] imply that the singularities of the function-field-case analog of certain affine Schubert varieties are known to correspond to those of the full nilpotent cone for G-conjugation action on $\mathfrak{g} = \mathrm{Lie}(G)$ where $G = \mathrm{GL}_n(k)$, and they are locally complete intersections and normal by the results on the nilpotent cones. Our methods are more direct as well as elementary, and are obtained independently from the result of the nilpotent cones in the affine Grassmannian associated with $G = \mathrm{SL}_n$.

The geometry of Y_r is indeed similar to that of the nilpotent variety, with Gorenstein and normal properties also holding true for the subregular nilpotent variety which is the complement of the dominant orbit by $\mathrm{GL}_n(k)$-conjugation action, and which consists of irregular nilpotent elements of a semisimple Lie algebra (see for example, [3,8]). For the precise connection between Y_r when $r = 1$ and the variety of p-*semilinear* nilpotent endomorphisms of k^n, we refer the reader to [7, Section 6]. Over the prime field $\mathbb{F}_p$, they are the nilpotent endomorphisms.

There have been other interests in $\mathbb{L}\mathrm{at}^n_r(K)$ (and its variations for other groups) as a parameter space of certain lattices, the objects that should be related to (the dual notion of) certain p-divisible groups. Moduli spaces of p-divisible groups appear, for example, in the mod p structure of a local model of some Shimura varieties (cf. [13]). Although $\mathbb{L}\mathrm{at}^n_r(K)$ arises as a positive characteristic structure which does not come from reduction mod p, studying its precise relevance to these local models should be an important investigation.

The proof of the Main Theorem follows the basic ideas discussed for the case Y_r in Section 2, and after reviewing what is known for Y_r, most of the rest of the paper is devoted to proving the analog of Proposition 2.1 for the covering scheme to the subregular variety, which will be defined and studied in Section 3. In Section 4 we prove the dimension formula of certain double coset closures, and it will be related to the covering scheme, namely, the analog of Proposition 2.1 is established, and in Section 5 the Main Theorem is proved.

2. Geometry of the full lattice variety: summary

We shall review, as the simplest case, the basic construction and properties of the matrix cover variety X_r and the morphism $\pi : X_r \to Y_r$. The proofs of the statements in this section are provided in [7,15] although some computations are repeated here. In [6,7], both X_r and Y_r were defined as schemes representing functors of points in the affine scheme Spec R for any k-algebra R. Note that in [7], X_r is denoted by $\mathcal{B}_{nr}(F)$.

We fix some notations first. Write the elements of $\mathcal{O}$ by

$$a = \sum_{i=0}^{\infty} \xi(a_i)^{p^{-i}} p^i$$

where $\xi : k \to \mathcal{O}$ is the canonical map, multiplicative on $k^\times$, which sections $\mathcal{O} \to k$. Let $\mathfrak{v} : \mathcal{O} \to \mathcal{O}$ be the Verschiebung morphism and extend it to $\mathfrak{v} : K \to K$ given by $x_i \mapsto x_{i-1}$ on all coordinates x_i with $x_i(b) = b_i$ where $b = \sum_{j=N}^{\infty} \xi(b_j)^{p^{-j}} p^j$ and $N \in \mathbb{Z}$. Define, for $F = \mathcal{O}^n$ and $u_j = \sum_{i=N}^{\infty} \xi(u_{i,j})^{p^{-i}} p^i \in K$, the morphism $\mathfrak{v}_F : K^n \to K^n$ by

$$\mathfrak{v}_F(u_1, \ldots, u_n)^T := (\mathfrak{v}(u_1), \ldots, \mathfrak{v}(u_n))^T.$$

Here, $(u_1, \ldots, u_n)^T$ denotes the transpose of the row vector $(u_1, \ldots, u_n)$. As in Section 1, let Y_r be identified, via $\mathfrak{v}^r_F$, with the set of lattices L such that $p^{nr}F \subset L \subset F$ with $\dim_k L/p^{nr}F = (n-1)nr$.

In what follows, we shall write, by abuse of notation,

$$\mu_r = \mathrm{diag}(p^{nr}, 1, \ldots, 1).$$

Denote by $\overline{\mathcal{O}} := \mathcal{O}/p^{nr+1}\mathcal{O}$ the finite Witt vectors of length $nr+1$, and write the elements of $\overline{\mathcal{O}}$ as $\sum_{i=0}^{nr} \xi(a_i)^{p^{-i}} p^i$ where p^i are understood to be the residue classes of p^i modulo $p^{nr+1}\mathcal{O}$.

Let $M_n(\overline{\mathcal{O}})$ be the affine k-space of dimension $n^2(nr+1)$ consisting of $n \times n$-matrices with entries in $\overline{\mathcal{O}}$. For $A = (a_{i,j}) \in M_n(\overline{\mathcal{O}})$, denote by S the set of variables

$$\{x_{i,j,s} \mid 1 \le i, j \le n, 0 \le s \le nr\}$$

for the coordinates of $a_{i,j,s}$ where

$$a_{i,j} = \sum_{s=0}^{nr} \xi(a_{i,j,s})^{p^{-s}} p^s,$$

and write $k[S]$ for the coordinate ring of $M_n(\overline{\mathcal{O}})$. Let G be the k-algebraic group $\mathrm{GL}_n(\overline{\mathcal{O}})$, and by $B \subset G$ the image of the standard upper triangular Iwahori subgroup of $\mathrm{GL}_n(\mathcal{O})$ under the map induced by the canonical map $\mathcal{O} \to \overline{\mathcal{O}}$. Then G and B are finite-dimensional algebraic groups over k. (The group G_0 still denotes the proalgebraic group $\mathrm{SL}_n(\mathcal{O})$.) Let

$$\Gamma(nr+1) = \{g \in \mathrm{GL}_n(\mathcal{O}) \mid g \equiv I \pmod{p^{nr+1}\mathcal{O}}\}$$

be the congruence subgroup. It acts trivially on Y_r, hence G, isomorphic to $\mathrm{GL}_n(\mathcal{O})/\Gamma(nr+1)$, acts algebraically on Y_r, reducing the infinite-dimensional nature of our problems to the ordinary algebraic geometry of finite-dimensional (non-reductive) algebraic group G over k. The action of $G \times G$ on $M_n(\overline{\mathcal{O}})$ defined by $((g,h), M) \mapsto gMh^{-1}$, where $(g,h) \in G \times G$, $M \in M_n(\overline{\mathcal{O}})$, is k-algebraic.

Let $\det : M_n(\overline{\mathcal{O}}) \to \overline{\mathcal{O}}$ be the determinant morphism defined for all $A \in M_n(\overline{\mathcal{O}})$, by taking $\det A \in \mathcal{O}$ and its canonical image in $\overline{\mathcal{O}}$. By the general properties of Witt schemes, for all $j = 0, 1, \ldots$, there exist polynomials $\delta_j \in k[S]$ so that for all $A \in M_n(\overline{\mathcal{O}})$,

$$\det A = \sum_{j \geq 0} \xi(\delta_j(A))^{p^{-j}} p^j.$$

We shall consider these polynomials with coefficients in an arbitrary k-algebra R. Then for $A \in M_n(\overline{\mathcal{O}})$, $\det A = p^{nr}u$ for some unit $u \in \overline{\mathcal{O}}$ if and only if

$$\delta_0(A) = \delta_1(A) = \cdots = \delta_{nr-1}(A) = 0, \quad \text{and} \quad \delta_{nr}(A) \in R^{\times}.$$

If $\mathfrak{a}$ is the ideal in $R[S]_{\delta_{nr}} = R \otimes_k k[S]_{\delta_{nr}}$ generated by $\delta_0, \ldots, \delta_{nr-1}$, let

$$X_r(R) := \mathrm{Spec}(R[S]_{\delta_{nr}}/\mathfrak{a}).$$

As a set of points in $\mathrm{Spec}\, k$,

$$X_r := X_r(k) = \{A \in M_n(\overline{\mathcal{O}}) \mid \det A = p^{nr}u \text{ for some } u \in \overline{\mathcal{O}}^{\times}\}.$$

Thus, X_r is a closed subscheme of the principal affine open set

$$D(\delta_{nr}) = \mathrm{Spec}(k[S]_{\delta_{nr}}) \subset M_n(\overline{\mathcal{O}}).$$

Then

Proposition 2.1 ([7,15]). *The scheme X_r is an affine variety which is a complete intersection in $M_n(\overline{\mathcal{O}})$.*

We will not prove the proposition here, but we indicate some of its main points. In any case, we establish the complete intersection property of X_r by computing its dimension using its orbit structure. More precisely, from [15], one checks that

$$X_r \subset \bigcup_{\substack{\eta \preceq \mu_r \\ \eta \in \Gamma^+}} G\eta G \subset \overline{G\mu_r G} \tag{2.1}$$

with the closure in $D(\delta_{nr})$. The first inclusion in (2.1) is a version of the Cartan decomposition of elements in X_r. We show this by an elementary row and column operations over $\overline{\mathcal{O}}$.

Proposition 2.2. *If $A \in X_r$, then $A \in G\gamma G$ for some $\gamma \in \Gamma^+$ such that $\gamma \preceq \mu_r$.*

Proof. Let $A = (a_{i,j}) \in X_r$ and look for the non-zero entry $a_{i,j} = p^{r_n}u$ with the smallest value r_n among the entries of A and u a unit in $\overline{\mathcal{O}}$. Locate the entry $a_{i',j} = p^{r_n}u'$ in the jth column with the largest row index $i' \geq i$ with the value r_n is achieved. Each non-zero entry $a_{k,j} = p^s v$ with $k > i'$ has strictly larger value than r_n, $s > r_n$. By left multiplying the column by an elementary matrix of the form $I - p^{s-r_n}u^{-1}vE_{k,i'}$, where $E_{k,i'}$ is the matrix with 1 in (k, i')-position and 0 everywhere else. Repeat this to clear all non-zero entries below $a_{i',j} = p^{r_n}u'$. For non-zero entries above it, the values, t are greater than or equal to r_n. Left multiply the column by $I - p^{t-r_n}u^{-1}vE_{\ell,i'}$ to clear $a_{\ell,j} = p^t v$. Repeat this so that the jth column of A has only one non-zero entry $a_{i',j}$. Now look for the entry with the smallest value $r_{n-1} \geq r_n$ in the rest of the columns and repeat the process, until one obtains the formula $UA = M$ where U is a product of elementary matrices and M a monomial matrix. Apply permutation matrices σ, τ and a diagonal matrix D with unit entries so that $\sigma UA\tau D = \sigma M\tau D = \mathrm{diag}(p^{r_1}, \ldots, p^{r_n}) = \gamma$ is a dominant cocharacter. Hence $A \in G\gamma G$. $\square$

The second inclusion in (2.1) is shown, for example, by a type of degeneration argument similar to [6, Proposition 7 (2)]. This will be used in the following sections, hence we prove it as follows.

Proposition 2.3 (*Degeneration of Cocharacters*). *If $\gamma \preceq \gamma'$ are two dominant cocharacters, then $G\gamma G \subset \overline{G\gamma' G}$.*

Proof. Consider an arbitrary pair (γ, γ') of cocharacters in Γ such that γ is dominant and

$$\gamma = \mathrm{diag}(p^{r_1}, \dots, p^{r_{i-1}}, p^{r_i}, p^{r_{i+1}}, \dots, p^{r_n}),$$
$$\gamma' = \mathrm{diag}(p^{r_1+b}, \dots, p^{r_{i-1}}, p^{r_i-b}, p^{r_{i+1}}, \dots, p^{r_n})$$

with $b > 0$ such that $\gamma \preceq \gamma'$. If γ' is not dominant, it is still in the same $G \times G$-orbit of the dominant cocharacter obtained by a pair of permutation matrices multiplied on the left and right of γ' so that $r_{i-1} \geq r_i - b \geq r_{i+1}$. Hence we may assume both γ and γ' above are dominant. We need to show that

$$G\gamma G \subset \overline{G\gamma' G}. \tag{2.2}$$

We want to degenerate γ to a family parametrized by $t \in k$ so that the family lies inside $G\gamma' G$, that is, each member of the family for all $t \in k^\times$ factors as

$$\gamma(t) = x\gamma' y^{-1}, \tag{2.3}$$

for some $x, y \in G$. Suppose a family of matrices parametrized by $k^\times$ is constructed in $G\gamma' G$ as below:

$$\mathcal{F} = \left\{ \gamma(t) := \left. \begin{pmatrix} p^{r_1} & & & & \\ & \ddots & & & \\ p^{r_i-b}\xi(t) & & p^{r_i} & & \\ & & & \ddots & \\ & & & & p^{r_n} \end{pmatrix} \right| t \in k^\times \right\} \subset G\gamma' G. \tag{2.4}$$

Then $\gamma = \gamma(0) \in \overline{G\gamma' G}$, hence (2.2). It then suffices to check (2.3) for $\gamma(t)$ given in (2.4).

Consider the embedding $j_{D,D'} : M_2(\overline{\mathcal{O}}) \to M_n(\overline{\mathcal{O}})$ defined by two diagonal matrices D, D'

$$j_{D,D'}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & & b & \\ & D & & \\ c & & d & \\ & & & D' \end{pmatrix}.$$

The map $j_{D,D'}$ preserves the matrix multiplication in $M_2(\overline{\mathcal{O}})$, so by this embedding, we can reduce (2.3), to the corresponding equation for the 2×2 matrix

$$\begin{pmatrix} p^{nr-d} & 0 \\ p^{c-b}\xi(t) & p^c \end{pmatrix} \in M_2(\overline{\mathcal{O}}),$$

that is, showing (2.3) is equivalent to showing

$$\begin{pmatrix} p^{nr-d} & 0 \\ p^{c-b}\xi(t) & p^c \end{pmatrix} = (x_{i,j}) \begin{pmatrix} p^{nr-d+b} & 0 \\ 0 & p^{c-b} \end{pmatrix} (y_{i,j}), \tag{2.5}$$

for some $(x_{i,j}), (y_{i,j}) \in \mathrm{GL}_2(\overline{\mathcal{O}})$. (Take $D = \mathrm{diag}(p^{r_2}, \dots, p^{r_{i-1}})$ and $D' = \mathrm{diag}(p^{r_{i+1}}, \dots, p^{r_n})$.) For all $t \neq 0$, $\xi(t)^{-1} = \xi(t^{-1}) \in \overline{\mathcal{O}}^\times$, and the following factorization formula checks (2.5)

$$\begin{pmatrix} p^{nr-d} & 0 \\ p^{c-b}\xi(t) & p^c \end{pmatrix} = \begin{pmatrix} 1 & -p^{nr-d+b-c}(p^b-1)\xi(t)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{nr-d+b} & 0 \\ 0 & p^{c-b} \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 \\ \xi(t) & 1 \end{pmatrix} \begin{pmatrix} 1 & (p^b-1)\xi(t)^{-1} \\ 0 & 1 \end{pmatrix}. \quad \square$$

Let $g, h \in G$. Then for all $A \in X_r$, $gAh^{-1} \in X_r$ by the definition of X_r. In particular, $G\mu_r G \subset X_r$, and hence with the inclusions in (2.1),

$$X_r = \overline{G\mu_r G}.$$

We use this equality to compute $\dim X_r$. It suffices to compute the dimension of the stabilizer $\mathrm{Stab}_{G\times G}(\mu_r)$ of μ_r under $G \times G$-action, so that

$$\dim G\mu_r G = \dim G \times G - \dim \mathrm{Stab}_{G\times G}(\mu_r),$$

which equals $\dim M_n(\overline{\mathcal{O}}) - nr$. (This is Proposition 4.1 with $\gamma = \mu_r$ in Section 4.) Hence X_r is a complete intersection in $M_n(\overline{\mathcal{O}})$, and being an orbit closure, the degeneration argument of Proposition 2.3 assures that all the $G \times G$-orbit closures are in the closure of the dominant orbit which is X_r.

We also showed in [7] that X_r is generically rational, hence reduced in particular. Rational parametrization of its dense open subset is given in [7] as a quotient of an affine space. This establishes the main argument of Proposition 2.1.

Last statements necessary to make the above construction work, are the following results from [7] (or [15, Section 4.2]), and from [1].

Proposition 2.4. *The morphism*

$$\pi : X_r \to Y_r$$

defined by $A \mapsto AF$ is surjective and smooth. In particular, π is faithfully flat.

Theorem 2.5 (*[1]*). *If $f : X \to Y$ is a faithfully flat morphism of schemes, X is a locally complete intersection if and only if Y and its fibers are also.*

(See [11, Remark after Theorem 23.6] which refers to [1].) By the theorem, Y_r is also a locally complete intersection. By Theorem 1.1 (3), Y_r is nonsingular in codimension one. By the Serre criterion for normality,

Theorem 2.6. *[7,15] The variety Y_r is normal and a locally complete intersection.*

Since X_r is also nonsingular in codimension one by the smoothness and surjectivity of π, it is also normal by the Serre criterion. The fiber of π at every point can be checked to be a right G-orbit, hence π is an orbit morphism. By another well-known theorem (cf. [2, Proposition 6.22]), the normality of Y_r implies that

Corollary 2.7 (*[7,15]*). *The variety Y_r is the geometric quotient of X_r by G, that is, π is a quotient morphism in the sense of G.I.T. (cf. [2,12]), and*

$$Y_r \simeq X_r/G = \overline{G\mu_r G}/G.$$

Remark 2.8. By the corollary, we can identify the closed subscheme $Y(\eta) = \overline{G_0\eta F}$ of Y_r, for all such $\eta \preceq \mu_r$, with $\overline{G\eta G}/G$ as follows. The orbit closure $\overline{G\eta G}$ in X_r is reduced and it is saturated for the surjection, that is, for any $z \in \overline{G\eta G}$, the fiber $\pi^{-1}(\pi(z)) \subset \overline{G\eta G}$. Thus its image under the quotient morphism π is closed, and it is equal to $Y(\eta)$. Hence $\overline{G\eta G} = \pi^{-1}(Y(\eta))$, and it is smooth over $Y(\eta) \cong \overline{G\eta G}/G$.

3. The scheme $X(i)$

In this section, we will define the scheme $X(i)$ and study its properties. We shall introduce notations and proceed to the main result of the section.

Define the k-algebraic subgroup B (resp. B^-) of $G = \mathrm{GL}_n(\overline{\mathcal{O}})$ as the image in G of the standard Iwahori subgroup of $\mathrm{GL}_n(\mathcal{O})$ whose reduction modulo p is the upper (resp. lower) triangular Borel subgroup of $\mathrm{GL}_n(k)$.

Matrices in $M_n(\overline{\mathcal{O}})$ are generally denoted by capital letters and when appropriate, in blocks:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & A_{1,3} \\ a_{2,1} & a_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix},$$

where $A_{1,3}, A_{2,3}$ denote rows with $n-2$ entries, $A_{3,1}, A_{3,2}$ denote columns with $n-2$ entries and $A_{3,3}$ denotes an $(n-2)\times(n-2)$ matrix. For $n=2$, whenever convenient, we write $A \in M_2(\overline{\mathcal{O}})$ using the block matrix notation, with $A_{1,3} = A_{2,3} = 0, A_{3,1} = A_{3,2} = 0$ and $A_{3,3} = I$.

Let $J, J' \subset \{1, \ldots, n\}$ be two subsets of equal size and let:

$$\Delta_{(J;J')}(A) = \sum_{s=0}^{nr} \xi(\delta_{(J;J'),s}(A))^{p^{-s}} p^s$$

denote the determinant of the minor matrix of A formed by removing the rows in J and the columns in J'. If $J = \{i\}, J' = \{j\}$, write $\Delta_{(J;J')}(A)$ as:

$$\Delta_{i,j}(A) = \sum_{s=0}^{nr} \xi(\delta_{i,j,s}(A))^{p^{-s}} p^s$$

where $\{\delta_{i,j,s} \mid s = 0, 1, \ldots, nr\}$ are polynomials in $k[S]$. If $J = J' = \{1, 2\}$, we obtain $\Delta_{(1,2;1,2)}(A)$ as:

$$\det(A_{3,3}) = \sum_{s=0}^{nr} \xi(\beta_s(A))^{p^{-s}} p^s,$$

where $\{\beta_s \mid s = 0, 1, \ldots, nr\}$ are polynomials in $k[S]$.

For all i such that $0 \le i \le nr$, let us denote by

$$X(i) = \mathrm{Spec}(k[X_r]/\mathfrak{a}(i)),$$

where $\mathfrak{a}(i)$ is the ideal generated by $\{\delta_{1,1,j}, \delta_{2,2,\ell} \mid j, \ell = 0, 1, \ldots, i-1\}$, and $\mathfrak{a}(0) = \mathfrak{a}$. As the set of points in Spec k,

$$X(i) = \left\{A \in X_r \mid \Delta_{1,1}(A), \Delta_{2,2}(A) \in p^i\overline{\mathcal{O}}\right\}.$$

By definition, $X(i)$ is an affine subscheme of $M_n(\overline{\mathcal{O}})$ defined by vanishings of $nr + 2i$ polynomials

$$\delta_0, \dots, \delta_{nr-1}, \delta_{1,1,0}, \dots, \delta_{1,1,i-1}, \delta_{2,2,0}, \dots, \delta_{2,2,i-1}$$

and non-vanishing of δ_{nr}. Hence

$$\dim X(i) \geq n^2(nr+1) - (nr+2i).$$

The main result of this section is

Theorem 3.1. *Denote by $Z(\gamma)$ the scheme closure of $G\gamma G$ in X_r with $\gamma \preceq \mu_r$, and write $Z(i)$ for $Z(\gamma(i))$. Then*

$$X(1) \subset Z(1).$$

Proof. By Corollary 2.7 and Remark 2.8, $\pi : X_r \to Y_r$ is a geometric quotient morphism and $Z(\gamma(i)) = \pi^{-1}(Y(i))$ is smooth over $Y(i)$. Denote by $Z(i)$ the closure $Z(\gamma(i))$. The orbit decomposition given in [6, (3.10)]

$$\varinjlim_r \mathbb{L}\mathrm{at}_r^n(K) = \bigcup_{\gamma \in \Gamma^+} G_0 \gamma F$$

when restricted to $Y_r = X_r/G \simeq \mathbb{L}\mathrm{at}_r^n(K)$ where $X_r = \overline{G\mu_r G}$, gives the orbit decomposition

$$Y_r = \bigcup_{\substack{\gamma \preceq \mu_r \\ \gamma \in \Gamma^+}} G_0 \gamma F$$

so that

$$X_r = \pi^{-1}(Y_r) = \bigcup_{\substack{\gamma \preceq \mu_r \\ \gamma \in \Gamma^+}} \pi^{-1}(G_0 \gamma F) = \bigcup_{\substack{\gamma \preceq \mu_r \\ \gamma \in \Gamma^+}} G\gamma G. \tag{3.1}$$

The degeneration of cocharacters (Proposition 2.3) gives

$$\bigcup_{\substack{\gamma \preceq \gamma(i) \\ \gamma \in \Gamma^+}} G\gamma G \subset \overline{G\gamma(i)G} = Z(i),$$

hence for the theorem, it suffices to prove the inclusion

$$D(\beta_0) \subset \bigcup_{\substack{\gamma \preceq \gamma(1) \\ \gamma \in \Gamma^+}} G\gamma G$$

where $D(\beta_0)$ is the dense open subset of $X(1)$ in which $\det A_{3,3}$ is a unit. By (3.1), if $A \in X(1)$, then $A \in G\gamma G$ such that $\gamma \preceq \mu_r$ and γ is dominant.

Let $A \in D(\beta_0)$ and choose a factorization by (3.1) $A = X\gamma Y$ for some $X \in G$, $Y \in G$ and $\gamma \preceq \mu_r$ dominant, where X, Y are written as

$$\begin{pmatrix} x_{1,1} & x_{1,2} & X_{1,3} \\ x_{2,1} & x_{2,2} & X_{2,3} \\ X_{3,1} & X_{3,2} & X_{3,3} \end{pmatrix}, \qquad \begin{pmatrix} y_{1,1} & y_{1,2} & Y_{1,3} \\ y_{2,1} & y_{2,2} & Y_{2,3} \\ Y_{3,1} & Y_{3,2} & Y_{3,3} \end{pmatrix}$$

so that $X_{3,3}, Y_{3,3}$ can be chosen to be invertible. Write

$$\gamma = \begin{pmatrix} p^{r_1} & & \\ & p^{r_2} & \\ & & \gamma_{3,3} \end{pmatrix}$$

where $\gamma_{3,3} = \mathrm{diag}(p^{r_3}, \dots, p^{r_n})$ with $\sum r_i = nr$, and $r_1 \geq \dots \geq r_n \geq 0$.

Expand $A = X\gamma Y$ so that, if we ignore cases of letters,

$$A_{i,j} = p^{r_1} X_{i,1} Y_{1,j} + p^{r_2} Y_{i,2} Y_{2,j} + X_{i,3} \gamma_{3,3} Y_{3,j}.$$

Consider the invertible matrix

$$A_{3,3} = p^{r_1} X_{3,1} Y_{1,3} + p^{r_2} X_{3,2} Y_{2,3} + X_{3,3} \gamma_{3,3} Y_{3,3}.$$

If $r_2 = 0$, then $r_j = 0$ for all $j > 2$ for γ is dominant, so $r_2 + \dots + r_n = 0$, hence $r_1 = nr$ and $\gamma = \mu_r$, that is, $A \notin G\eta G$ for any $\eta \precneqq \mu_r$, a contradiction, hence $r_1 \geq r_2 \geq 1$. That is, $r_1 = nr - r_2 \leq nr - 1$, hence $\gamma \preceq \gamma(1)$, showing $D(\beta_0) \subset G\gamma G$, and so $X(1) = \overline{G\gamma G} \subset \overline{G\gamma(1)G}$. $\square$

Remark 3.2. It is a matter of computation to show the inclusion $G\gamma(i)G \subset X(i)$, although this is not needed in the proof of the Main Theorem. Clearly, $\gamma(i) \in X(i)$. Take $(X, Y) \in G \times G$. One can show that $\Delta_{1,1}(X\gamma(i)Y)$ and $\Delta_{2,2}(X\gamma(i)Y)$ are elements of $p^i\overline{\mathcal{O}}$ using notational help in matrix operations as seen in

Lemma 3.3. *Let A be a matrix such that $A_{3,3}$ is invertible. Then for all integers $i, j \in \{1, 2\}$ and $\ell \geq 0$, the determinant $\Delta_{\hat{i},\hat{j}}(A) \in p^\ell \overline{\mathcal{O}}$ if and only if $a_{i,j} - A_{i,3}A_{3,3}^{-1}A_{3,j} \in p^\ell\overline{\mathcal{O}}$, where $\hat{j}$ denotes the complement integer in $\{1, 2\}$ (that is, if $j = 1$, then $\hat{j} = 2$).*

Proof. Observe that

$$\begin{aligned} a_{i,j} - A_{i,3}A_{3,3}^{-1}A_{3,j} &= \det\begin{pmatrix} a_{i,j} - A_{i,3}A_{3,3}^{-1}A_{3,j} & A_{i,3}A_{3,3}^{-1} \\ 0 & I \end{pmatrix} \\ &= \det\begin{pmatrix} a_{i,j} & A_{i,3} \\ A_{3,j} & A_{3,3} \end{pmatrix} \det\begin{pmatrix} 1 & 0 \\ -A_{3,3}^{-1}A_{3,j} & A_{3,3}^{-1} \end{pmatrix} \\ &= \Delta_{\hat{i},\hat{j}}(A)\det A_{3,3}^{-1}. \quad \square \end{aligned}$$

By Lemma 3.3, it suffices to show

$$(X\gamma(i)Y)_{2,2} - (X\gamma(i)Y)_{2,3}(X\gamma(i)Y)_{3,3}^{-1}(X\gamma(i)Y)_{3,2},$$
$$(X\gamma(i)Y)_{1,1} - (X\gamma(i)Y)_{1,3}(X\gamma(i)Y)_{3,3}^{-1}(X\gamma(i)Y)_{3,1}$$

are in $p^i\overline{\mathcal{O}}$. We compute $\Delta_{1,1}$, and the case for $\Delta_{2,2}$ is carried out exactly the same way. Write

$$\begin{aligned} (X\gamma(i)Y)_{2,2} &= p^i m_{2,2} + X_{2,3}Y_{3,2}, \\ (X\gamma(i)Y)_{2,3} &= p^i M_{2,3} + X_{2,3}Y_{3,3}, \\ (X\gamma(i)Y)_{3,2} &= p^i M_{3,2} + X_{3,2}Y_{3,3} \end{aligned}$$

where

$$\begin{aligned} m_{2,2} &= p^{nr-2i}x_{2,1}y_{1,2} + x_{2,2}y_{2,2}, \\ M_{2,3} &= p^{nr-2i}x_{2,1}Y_{1,3} + x_{2,2}Y_{2,3}, \\ M_{3,2} &= p^{nr-2i}X_{3,1}y_{1,2} + X_{3,2}y_{2,2}. \end{aligned}$$

Also,

$$\begin{aligned} (X\gamma(i)Y)_{3,3}^{-1} &= (p^{nr-i}X_{3,1}Y_{1,3} + p^iX_{3,2}Y_{2,3} + X_{3,3}Y_{3,3})^{-1} \\ &= \left((p^iM + I)X_{3,3}Y_{3,3}\right)^{-1} \\ &= (X_{3,3}Y_{3,3})^{-1}(-p^iW + I) \end{aligned}$$

where $p^iM = p^i(p^{nr-2i}X_{3,1}Y_{1,3} + X_{3,2}Y_{2,3})(X_{3,3}Y_{3,3})^{-1}$ is a nilpotent $n-2 \times n-2$ matrix and W is such that

$$I - p^iW = I - p^iM + p^2iM^2 - \cdots = (p^iM + I)^{-1}.$$

Collecting all the terms divisible by p^i and denoting the sum by $p^in_{2,2}$ for some $n_{2,2} \in \overline{\mathcal{O}}$,

$$\begin{aligned} (X\gamma(i)Y)_{2,3}(X\gamma(i)Y)_{3,3}^{-1}(X\gamma(i)Y)_{3,2} &= (p^iM_{2,3} + X_{2,3}Y_{3,3})(X_{3,3}Y_{3,3})^{-1}(-p^iW + I)(p^iM_{3,2} + X_{3,3}Y_{3,2}) \\ &= p^in_{2,2} + X_{2,3}Y_{3,2}. \end{aligned}$$

Thus

$$\begin{aligned} (X\gamma(i)Y)_{2,2} - (X\gamma(i)Y)_{2,3}(X\gamma(i)Y)_{3,3}^{-1}(X\gamma(i)Y)_{3,2} &= p^im_{2,2} + X_{2,3}Y_{3,2} - (p^in_{2,2} + X_{2,3}Y_{3,2}) \\ &= p^i(m_{2,2} - n_{2,2}). \end{aligned}$$

4. The dimension of $X(1)$

The scheme $Z(\gamma) = \overline{G\gamma G}$ is irreducible and we showed that $X(1) \subset Z(1)$. In this section, we compute $\dim Z(\gamma)$. We will then show that $Z(1)$ is generically rational, hence reduced in particular, by quoting Rosenlicht's Theorem.

Recall that for a dominant cocharacter γ such that $Y(\gamma) \subset Y_r$, $Z(\gamma) = \pi^{-1}(Y(\gamma))$ and that π restricted to $Z(\gamma)$ is smooth over $Y(\gamma)$. Denote γ such that $\gamma \preceq \mu_r$ and written as the diagonal matrix $\mathrm{diag}(p^{r_1}, \ldots, p^{r_n})$ where $r_1 \geq \cdots \geq r_n \geq 0$ and $\sum r_i = nr$.

Proposition 4.1. *The following equality holds:*

$$\dim Z(\gamma) = n^2(nr+1) - \left(nr + 2\sum_{i=2}^{n}(i-1)r_i\right).$$

Proof. In the proof of Theorem 1.1 (2) which appears in [6] (see also Proposition 5.2 in [7]), it is shown that the dimension of G_0-dominant orbit of $Y(\gamma)$ (identified as the subscheme of $\mathbb{L}\mathrm{at}_r^n(K)$) is equal to $\sum_{i<j}(r_i - r_j)$. Reworking the computation of the sum for $r_i \geq 0$ and $\sum r_i = nr$, one observes that the sum contains r_i with positive one coefficient $n-i$ times and

with negative one coefficient $i-1$ times. Thus

$$\begin{aligned}\dim Y(\gamma) &= \sum_{i<j}(r_i - r_j)\\ &= \sum_{i=1}^{n}(n-i)r_i - \sum_{i=1}^{n}(i-1)r_i\\ &= \sum_{i=1}^{n}(n+1-2i)r_i\\ &= (n+1)nr - 2\sum_{i=1}^{n} ir_i.\end{aligned}$$

Put $r_1 = nr - \sum_{i=2}^{n} r_i$, so that

$$\begin{aligned}\dim Y(\gamma) &= (n+1)nr - 2\left(nr - \sum_{i=2}^{n}(1-i)r_i\right)\\ &= (n-1)nr - 2\sum_{i=2}^{n}(i-1)r_i\\ &= \dim Y_r - 2\sum_{i=2}^{n}(i-1)r_i.\end{aligned}$$

Consider the fiber over an arbitrary lattice $L \in Y(\gamma)$ of the smooth morphism $\pi|_{Z(\gamma)}$. Then the fiber is isomorphic as a scheme to a coset ηG where $G\eta F \subset Y(\gamma)$. The dimension of the fiber ηG for $\eta = \mathrm{diag}(p^{t_1}, \ldots, p^{t_n})$ is computed by observing that $A = (a_{i,j}) \in \mathrm{Stab}_G(\eta)$ if and only if $\eta A = \eta$, that is,

$$p^{t_i} a_{i,j} = p^{t_i}\delta_{i,j}$$

or $a_{i,j} \in \delta_{i,j} + p^{nr+1-t_i}\overline{\mathcal{O}}$, so $\dim \mathrm{Stab}_G(\eta) = \sum_i nt_i = n^2 r$, and that $\dim \eta G = n^2(nr+1) - n^2 r$.

By the smoothness (hence flatness) of π restricted to $Z(\gamma)$,

$$\begin{aligned}\dim Z(\gamma) &= \dim Y(\gamma) + \dim \pi^{-1}(L)\\ &= (n-1)nr + n^2(nr+1) - n^2 r - 2\sum_{i=2}^{n}(i-1)r_i\\ &= n^2(nr+1) - \left(nr + 2\sum_{i=2}^{n}(i-1)r_i\right). \quad \square\end{aligned}$$

It follows immediately the following

Corollary 4.2. *For all* $\gamma(i)$,

$$\dim Z(i) = n^2(nr+1) - (nr+2i).$$

Proof. Apply Proposition 4.1 for $\gamma(i)$, where $r_1 = nr - i$, $r_2 = i$, and $r_j = 0$ for all $j > 2$. $\square$

Remark 4.3. In the proof, the dimension formula

$$\dim Y(\gamma) = \dim Y_r - 2\sum_{i=2}^{n}(i-1)r_i$$

agrees with the length formula for $\ell(\gamma)$ known for affine Weyl groups in general. That is, if $2\rho^\vee$ denotes the dual of the sum of the positive roots, identified with the coordinate vector $(l, l-2, \ldots, -l) = \sum_{i<j} e_i - e_j$, with respect to the chosen basis $\{e_i \mid 1 \le i \le l+1\}$ given for the standard discussion of the root system A_l, and if $\langle\, ,\, \rangle$ denotes the dot product, then $n = l+1$ and

$$\begin{aligned}\ell(\gamma) &= \langle 2\rho^\vee, \gamma\rangle\\ &= (n-1)r_1 + (n-3)r_2 + \cdots + (-n+1)r_n\\ &= (n-1)r_1 + (n-1-2)r_2 + \cdots + (n-1-2(n-1))r_n)\\ &= (n-1)\sum_i r_i - 2\sum_{i=2}^{n}(i-1)r_i\\ &= \ell(\mu_r) - 2\sum_{i=2}^{n}(i-1)r_i.\end{aligned}$$

Recall the subgroups B, B^- of G, the canonical images in G of standard Iwahori subgroups. Since $Z(\gamma)$ is $G \times G$-stable, it is $B \times B^-$-stable by the restricted action. Moreover,

Proposition 4.4. *The following equality holds for dominant cocharacters γ such that $\gamma \preceq \mu_r$:*

$$\dim B\gamma B^- = n^2(nr+1) - \left(nr + 2\sum_{i=2}^{n}(i-1)r_i\right) = \dim G\gamma G. \tag{4.1}$$

Proof. We shall compute the dimension of the stabilizer $\mathrm{Stab}_{B\times B^-}(\gamma)$. Write $A = (a_{i,j}) \in B$, $C = (c_{i,j}) \in B^-$. By definition, $(A, C) \in B \times B^-$ stabilizes γ if and only if $A\gamma = \gamma C$, that is,

$$(p^{r_j}a_{i,j}) = (p^{r_i}c_{i,j})$$

holds as a matrix equality. For $i = j$, if $a_{i,i} \in \overline{\mathcal{O}}$, then $p^{r_i}(a_{i,i} - c_{i,i}) = 0$, hence $c_{i,i} = a_{i,i} + p^{nr+1-r_i}c'_{i,i}$ for some $c'_{i,i} \in \overline{\mathcal{O}}$. If $i > j$, then $r_j \geq r_i$ and $a_{i,j} \in p\overline{\mathcal{O}}$. Take such $a_{i,j}$ arbitrarily so that $p^{r_i}(c_{i,j} - p^{r_j-r_i}a_{i,j}) = 0$, hence $c_{i,j} = p^{r_j-r_i}a_{i,j} + p^{nr+1-r_i}c'_{i,j}$. Similarly, if $i < j$, then $r_i \geq r_j$ and take $c_{i,j} \in p\overline{\mathcal{O}}$ arbitrarily to get $a_{i,j} = p^{r_i-r_j}c_{i,j} + p^{nr+1-r_j}a'_{i,j}$. Then $\mathrm{Stab}_{B\times B^-}(\gamma)$ is a scheme isomorphic to the scheme of matrix elements of the form

$$\left(\begin{pmatrix} * & a_{i,j} \\ * & * \end{pmatrix}, \begin{pmatrix} c_{i,i} & * \\ c_{i,j} & c_{j,j} \end{pmatrix}\right) \in B \times B^-$$

where $\begin{pmatrix} * & * \\ * & * \end{pmatrix} \simeq B \cap B^-$ is of dimension equal to $\dim B - (n-1)n/2$, and the scheme of points consisting of

$$a_{i,j} \in p^{r_i-r_j}c_{i,j} + p^{nr+1-r_j}\overline{\mathcal{O}} \quad \text{for } i < j,$$
$$c_{i,j} \in p^{r_j-r_i}a_{i,j} + p^{nr+1-r_i}\overline{\mathcal{O}} \quad \text{for } i \geq j$$

whose total dimension is $\sum_i r_i + 2\sum_{i<j} r_j$. Hence

$$\begin{aligned}\dim \mathrm{Stab}_{B\times B^-}(\gamma) &= \dim B - (n-1)n/2 + \sum_i r_i + 2\sum_{i<j} r_j \\ &= \dim B - (n-1)n/2 + nr + 2\sum_{i=2}^{n}(i-1)r_i\end{aligned}$$

where, in $\sum_{i<j} r_j$, r_i appears $i-1$ times, for all $i > 1$, hence $\sum_{i<j} r_j = \sum_{i=2}^{n}(i-1)r_i$. Then

$$\begin{aligned}\dim B\gamma B^- &= 2\dim B - \left(\dim B - (n-1)n/2 + nr + 2\sum_{i=2}^{n}(i-1)r_i\right) \\ &= \dim B + (n-1)n/2 - \left(nr + 2\sum_{i=2}^{n}(i-1)r_i\right) \\ &= \dim G - \left(nr + 2\sum_{i=2}^{n}(i-1)r_i\right) \\ &= n^2(nr+1) - \left(nr + 2\sum_{i=2}^{n}(i-1)r_i\right). \quad \square\end{aligned}$$

An immediate consequence is

Corollary 4.5. *The scheme $Z(\gamma)$ is equal to $\overline{B\gamma B^-}$.*

Proof. Since $B\gamma B^- \subset G\gamma G$, and their dimensions are equal by comparing Corollary 4.2 and Proposition 4.4. $\square$

The action of the solvable linear group $B \times B^-$ on $Z(\gamma)$ is algebraic, so we can use the following result of Rosenlicht on rational cross sections for varieties with a solvable linear algebraic group action.

Theorem 4.6 ([14]). *Let k be a field, G a k-linear solvable algebraic group and V a transformation space for G (G-variety), all rational over k. Then there exists a G-invariant dense k-open subset V' of V such that V'/G exists and is rational over k and a cross section k-morphism $V'/G \to V'$ exists.*

Using this, we obtain

Theorem 4.7. *The scheme $Z(\gamma)$ is generically rational. In particular, it is reduced.*

Proof. By Theorem 4.6 applied to the homogeneous space $V = B\gamma B^-$, there exists a k-rational point in V. □

The following analog of Proposition 2.1 relating $X(1)$ and $Z(1)$ is the main ingredient of the proof of the Main Theorem.

Theorem 4.8. *The schemes $X(1)$ and $Z(1)$ are equal, hence $X(1)$ is a locally complete intersection variety in the affine $n^2(nr+1)$-space $M_n(\overline{\mathcal{O}})$, defined by $nr+2$ polynomials.*

Proof. By Theorem 3.1,

$$X(1) \subset \overline{G\gamma(1)G} = Z(1)$$

hence by Corollary 4.2, $\dim X(1) \leq \dim Z(\gamma(1)) = n^2(nr+1) - (nr+2)$. By definition, $X(1)$ is the scheme defined by $nr+2$ polynomials in the affine $n^2(nr+1)$-space, hence $\dim X(1) = n^2(nr+1) - (nr+2)$. Hence $X(1)$ is a closed, reduced (since $Z(1)$ is generically rational) and irreducible subscheme which is a complete intersection in a nonsingular subvariety $D(\delta_{nr})$ of $M_n(\overline{\mathcal{O}})$. □

Remark 4.9. It is not known whether the inclusion $X(i) \subset Z(i)$ is true for $i > 1$.

5. Proof of the main theorem

With the covering scheme's desired properties shown, we prove the normality of Y_r by using the Serre Criterion. We need:

Lemma 5.1. *The scheme $Y(i)$ is regular in codimension one ('R1').*

Proof. The partial ordering, $\gamma(i+1) \prec \gamma(i)$, and the degeneration argument in [6, Proposition 7] imply that

$$Y(i+1) \subset Y(i).$$

The dimension computation in Theorem 1.1 (2) implies that

$$\dim Y(i+1) = \dim Y(i) - 2.$$

The complement of the smooth locus of $Y(i)$ is G_0-stable, hence it is a finite union of G_0-orbits, each of which is even dimensional. Hence the singular locus of $Y(i)$ is of codimension at least two. This shows $Y(i)$ is regular in codimension one. □

By Remark 2.8, the restriction of $\pi : X_r \to Y_r$ to $Z(\gamma)$ has the image $Y(\gamma)$ and the map is smooth. We take $i = 1$ for $\gamma = \gamma(i)$.

By Theorem 4.8, $X(1) = Z(1)$ is a locally complete intersection. By Theorem 2.5, $Y(1)$ is a k-variety which is a locally complete intersection. By Lemma 5.1, $Y(1)$ is nonsingular in codimension one. By the Serre Criterion, $Y(1)$ is normal, proving the Main Theorem.

Remark 5.2. As a byproduct, $X(1)$ is also 'R1' under π restricted, hence normal by the Serre criterion.

Remark 5.3. The equality (4.1) implies that

$$\overline{B\gamma G} \subset \overline{G\gamma G} = \overline{B\gamma B^-} \subset \overline{B\gamma G},$$

and by definition as well as Remark 2.8, that $\overline{B\gamma G}/G \simeq Y(\gamma) = \overline{B_0\gamma F}$ where B_0 is the upper triangular Iwahori subgroup of $\mathrm{SL}_n(\mathcal{O})$.

Remark 5.4. The Main Theorem, and more importantly, the methods used in the paper remain valid for the affine Schubert varieties $\overline{B_0\mu_r F}$ in the *true* affine Grassmannian $G(K)/G(\mathcal{O})$, where the local field K is in equal characteristic with its residue field k, that is, for the case of the algebraic loop group of $G = \mathrm{SL}_n$, without any change from our mixed characteristic case. The corresponding subregular variety is normal and a locally complete intersection.

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